Variable Eddington Factor Acceleration of Thermal Radiative Transfer on Curved Meshes

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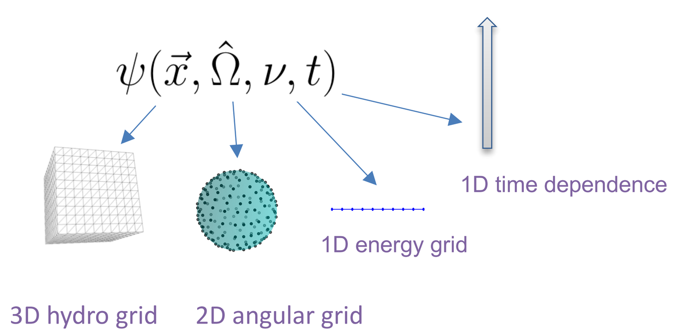
Motivation

- LDRD to investigate solving radiative transfer on curved, high order meshes with high order finite elements
- Curved meshes introduce non-trivial difficulties (face integration, sweep scheduling, reentrant faces)
- Iterative convergence rate is slow in optically thick limit
- Goal: design a method that improves iterative convergence while enabling curved mesh transport

Background

- Mono-energetic, steady state, Boltzman Transport Equation with isotropic scattering, source:

$$\hat{\Omega}\cdot\nabla\psi + \sigma_t\psi = \frac{\sigma_s}{4\pi}\int_{4\pi}\psi\,\mathrm{d}\Omega + \frac{Q}{4\pi}$$



- S_N angular discretization:

$$\hat{\Omega}_d\cdot\nabla\psi_d + \sigma_t\psi_d = \frac{\sigma_s}{4\pi}\sum_{d'} w_{d'}\psi_{d'} + \frac{Q}{4\pi}$$

where $\psi_d = \psi(\hat{\Omega}_d)$ and the $\hat{\Omega}_d$ are the discrete angles in a quadrature rule $\{\hat{\Omega}_d, w_d\}$

- Source Iteration: decouple in angle by lagging scattering term:

$$\hat{\Omega}_d\cdot\nabla\psi_d^{\ell+1} = \frac{\sigma_s}{4\pi}\phi^\ell + \frac{Q}{4\pi}$$

where $\phi^\ell = \sum w_d\psi_d^\ell$

- Convergence rate dependent on amount of scattering $\Rightarrow$ acceleration required

Variable Eddington Factor Acceleration

- Take angular moments of the transport equation

$$\nabla\cdot\vec{J} + \sigma_a\phi = Q\,,$$
$$\nabla\cdot\int\hat{\Omega}\hat{\Omega}\,\psi\,\mathrm{d}\Omega + \sigma_t\vec{J} = 0\,,$$

where $\phi = \int\psi\,\mathrm{d}\Omega$, $\vec{J} = \int\hat{\Omega}\,\psi\,\mathrm{d}\Omega$

- Multiply and divide by ϕ

$$\nabla\cdot\int\hat{\Omega}\hat{\Omega}\,\psi\,\mathrm{d}\Omega \to \nabla\cdot\underbrace{\frac{\int\hat{\Omega}\hat{\Omega}\,\psi\,\mathrm{d}\Omega}{\int\psi\,\mathrm{d}\Omega}}_{\text{Eddington Tensor}=\boldsymbol{E}}\phi$$

- Eddington Equations:

$$\nabla\cdot\vec{J} + \sigma_a\phi = Q\,,$$
$$\nabla\cdot\boldsymbol{E}\phi + \sigma_t\vec{J} = 0\,.$$

- Transport sweep for $\psi^{\ell+1/2}$ to compute $\boldsymbol{E}^{\ell+1/2}$

$$\boldsymbol{E}_{ij}^{\ell+1/2} = \frac{\sum\hat{\Omega}_i\hat{\Omega}_j\,\psi_d^{\ell+1/2}w_d}{\sum\psi_d^{\ell+1/2}w_d}$$

and solve

$$\nabla\cdot\vec{J} + \sigma_a\phi^{\ell+1} = Q\,,$$
$$\nabla\cdot\boldsymbol{E}^{\ell+1/2}\phi^{\ell+1} + \sigma_t\vec{J} = 0\,.$$

for the updated scalar flux, $\phi^{\ell+1}$

- If not converged, update the S_N scattering term and repeat until $\phi^{\ell+1}$ and ϕ^ℓ converge

Mixed Finite Element Discretization

- Discretize $\vec{J}$ with $H^{1,d}$ elements and ϕ with L^2

$$\vec{J}\approx\vec{J}_h = \sum\vec{S}_iJ_i\,,\quad \vec{S}_i\subset H^{1,d}$$
$$\phi\approx\phi_h = \sum B_i\phi_i\,,\quad B_i\subset L^2$$

- Multiply zeroth moment by ϕ basis function and integrate globally

$$\int B_i\nabla\cdot\vec{J}_h\,\mathrm{d}V + \int\sigma_aB_i\phi_h\,\mathrm{d}v = \int B_iQ\,\mathrm{d}V$$

- Multiply first moment by $\vec{J}$ basis function and integrate globally (with Gauss Divergence Theorem to offload gradient of L^2)

$$\int\phi_h\boldsymbol{E}:\vec{S}_i\,\mathrm{d}V - \int\sigma_t\vec{S}_i\cdot\vec{J}_h\,\mathrm{d}V = \int_{\partial V}\boldsymbol{E}\hat{n}\cdot\vec{S}_i\phi_h^{\mathrm{BC}}\,\mathrm{d}S$$

- Leads to non-symmetric Saddle Point system:

$$\begin{bmatrix}\mathbf{M}_a & \mathbf{G}\\ \mathbf{H} & -\mathbf{M}_t\end{bmatrix}\begin{bmatrix}\underline{\phi}\\ \underline{J}\end{bmatrix} = \begin{bmatrix}\underline{Q}\\ \underline{B}\end{bmatrix}$$

where

$$\mathbf{M}_{a,ij} = \int\sigma_aB_iB_j\,\mathrm{d}V\,,\qquad \mathbf{G}_{ij} = \int B_i\nabla\cdot\vec{S}_j\,\mathrm{d}V\,,$$
$$\mathbf{H}_{ij} = \int B_j\boldsymbol{E}:\nabla\vec{S}_i\,\mathrm{d}V\,,\qquad \mathbf{M}_{t,ij} = \int\sigma_t\vec{S}_i\cdot\vec{S}_j\,\mathrm{d}V$$
$$\underline{Q}_i = \int B_iQ\,\mathrm{d}V\,,\qquad \underline{B}_i = \int_{\partial V}\boldsymbol{E}\hat{n}\cdot\vec{S}_i\phi_h^{\mathrm{BC}}\,\mathrm{d}S$$

Schur Complement Solve

- $\mathbf{M}_a$ is block diagonal $\Rightarrow$ easily inverted!

$$\mathbf{M}_a\underline{\phi} + \mathbf{G}\underline{J} = \underline{Q}$$
$$\Rightarrow\underline{\phi} = \mathbf{M}_a^{-1}\left[\underline{Q} - \mathbf{G}\underline{J}\right]$$
$$\mathbf{H}\underline{\phi} - \mathbf{M}_t\underline{J} = \underline{B}$$
$$\Rightarrow -\underbrace{\left[\mathbf{H}\mathbf{M}_a^{-1}\mathbf{G} + \mathbf{M}_t\right]}_{\text{Schur Complement}=\mathbf{S}}\underline{J} = \underline{B} - \mathbf{H}\mathbf{M}_a^{-1}\underline{Q}$$

- Assemble and solve at every iteration $\Rightarrow$ want iterative solver
- $\mathbf{S}$ is non-symmetric and has been difficult to solve iteratively (need a preconditioner)

Results

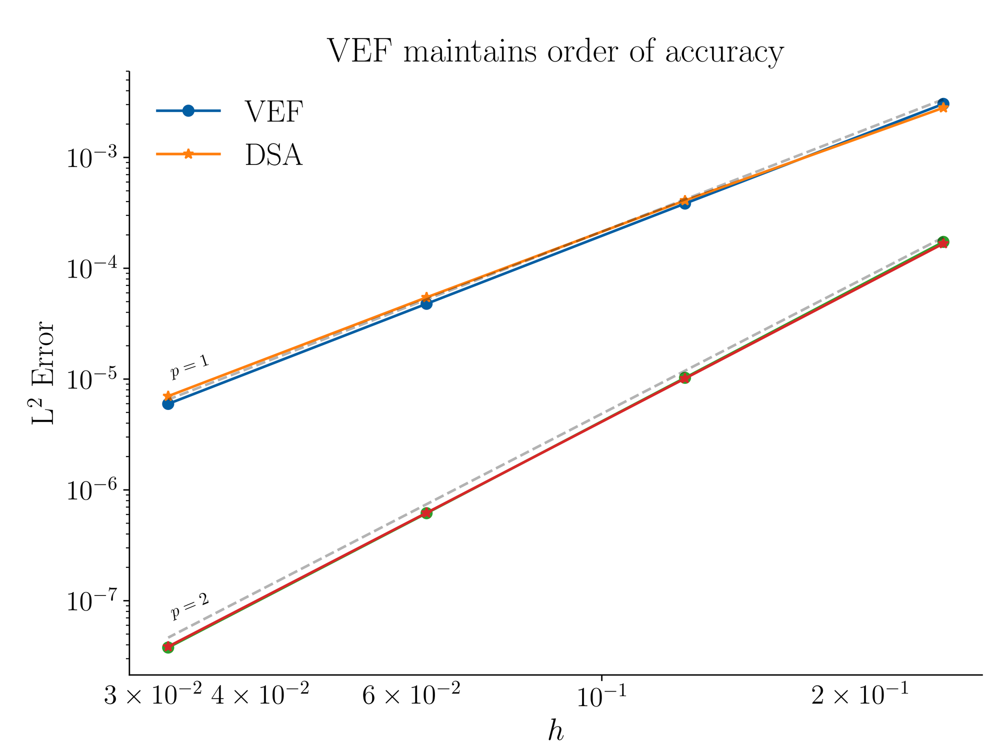


Fig. 1: Method of Manufactured Solutions error compared to reference third and fourth order lines.

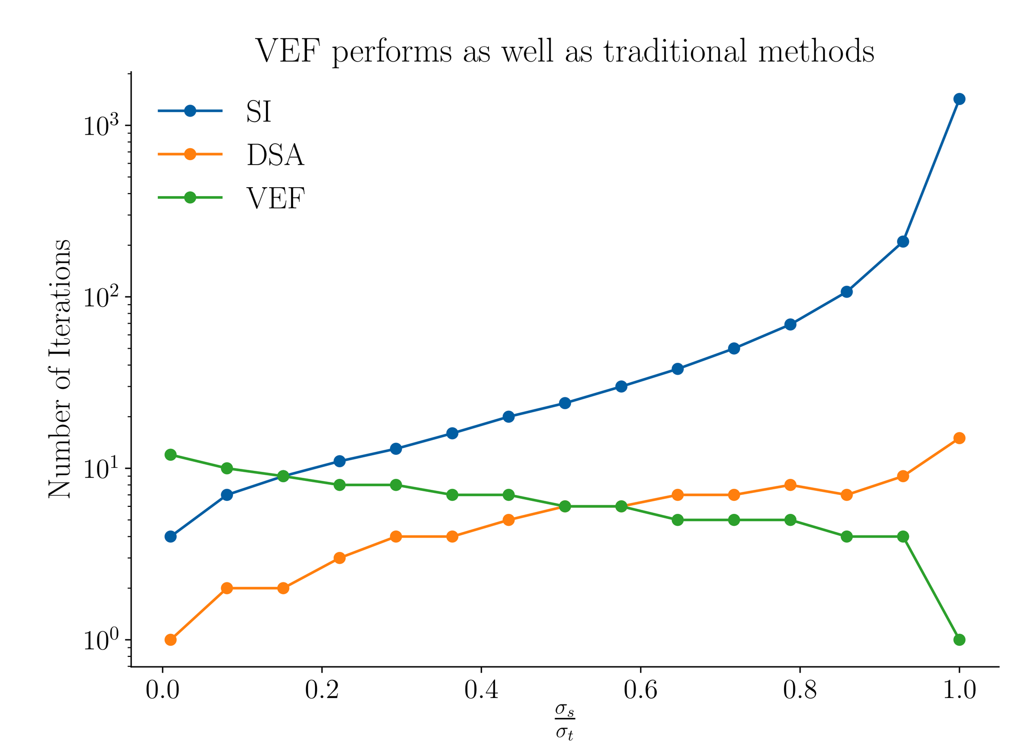


Fig. 2: Number of iterations required for convergence to 10^{-8} for a range of scattering ratios.

Two Mesh Approach

- Alleviate curved mesh issues by sweeping on simpler, low order mesh. VEF on original high order hydro mesh
- Map $\boldsymbol{E}$ on linear mesh to curved mesh and scattering term on curved mesh to linear mesh

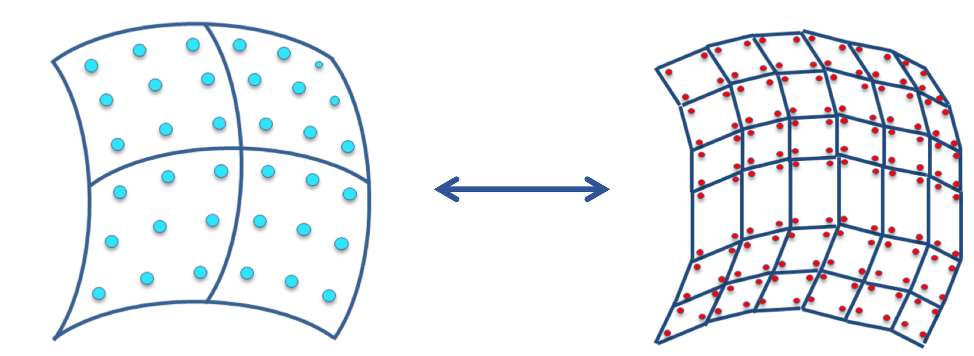


Fig. 3: Depiction of DOFs in the curved and straightened/refined meshes.

Conclusions and Future Work

- Developed an arbitrary order Mixed FEM VEF discretization
- Showed that VEF accelerates source iteration
- Future Work:
  - Design a preconditioner to iteratively solve the Schur Complement system
  - Investigate stability of transport on straightened mesh, VEF on curved mesh
  - Investigate using VEF as a preconditioner (VEFSA?)

References

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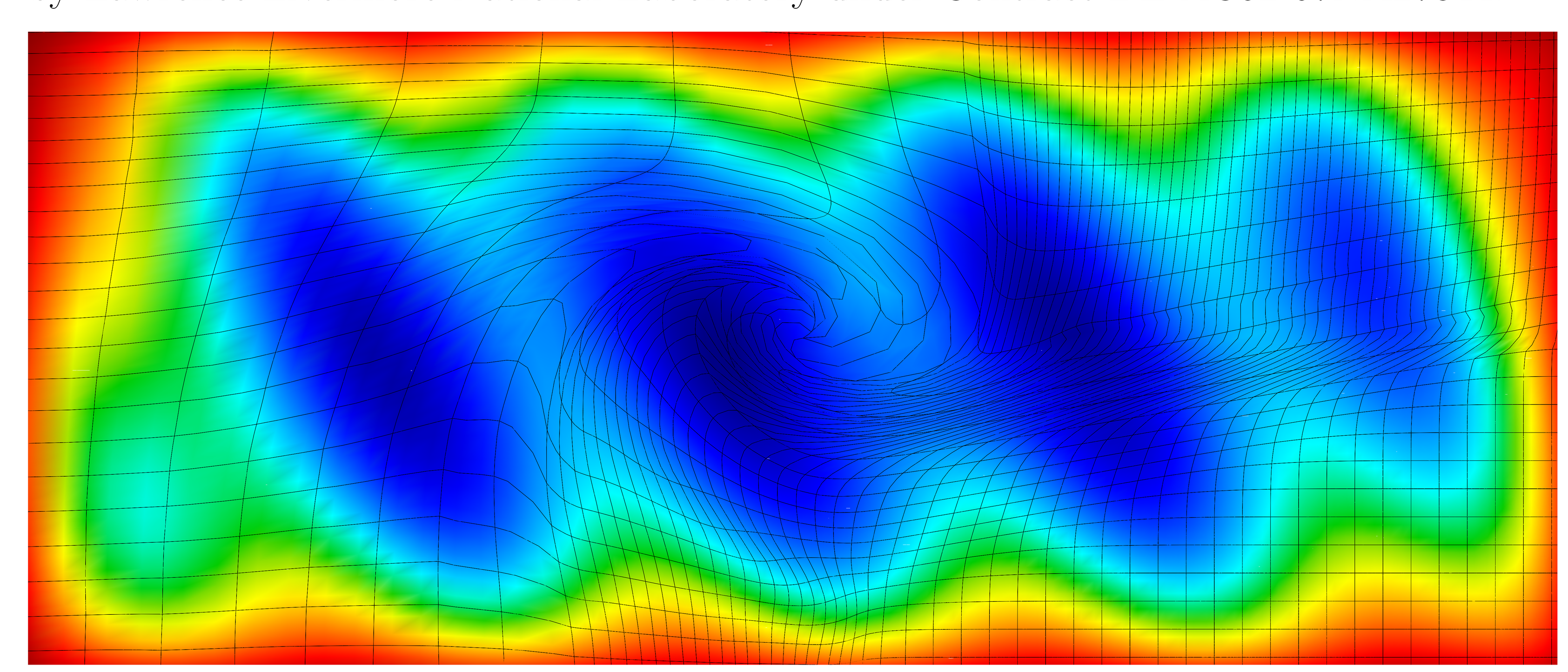

Fig. 4: Transport solve on BLAST's triple point mesh.

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