Matrix-Free Hyper-Viscosity Formulation for High-Order Lagrangian Hydrodynamics

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Abstract

Numerical approximation of shock hydrodynamics is at the core of multiphysics simulations. Artificial viscosity enables shock capturing by augmenting the hydrodynamics equations with an artificial term. For high-order simulations, smart artificial tensors need to turn themselves off in regions where the solution is smooth in order to preserve the convergence properties of the method. In this work, we present a high-order hyper viscous term that achieves shock capturing while remaining high-order. Furthermore, we propose an assembly-free implementation of the artificial viscous term that improves scalability and maximizes performance. Simulations were run with the Laghos miniapp in MFEM.

Hydrodynamics Equations

Inviscid gases are modeled by the *Euler-Equations*

Euler Equations

Continuous Form	Discrete Form	Type of Physics
$\rho\frac{\mathrm{d}\vec{\boldsymbol{v}}}{\mathrm{d}t} = \nabla\cdot\boldsymbol{\sigma}$	$\boldsymbol{M}_{\mathcal{V}}\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = -\boldsymbol{F}\cdot\mathbf{1}$	momentum conservation
$\rho\frac{\mathrm{d}e}{\mathrm{d}t} = \boldsymbol{\sigma}:\nabla\vec{\boldsymbol{v}}$	$\boldsymbol{M}_{\mathcal{E}}\frac{\mathrm{d}\boldsymbol{e}}{\mathrm{d}t} = \boldsymbol{F}^{\top}\cdot\boldsymbol{v}$	energy conservation
$\frac{\mathrm{d}\vec{\boldsymbol{x}}}{\mathrm{d}t} = \vec{\boldsymbol{v}}$	$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{v}$	equation of motion

where $\vec{\boldsymbol{x}}$ = position, $\vec{\boldsymbol{v}}$ = velocity, ρ = density, e = internal energy, and $\boldsymbol{\sigma} = -EOS\,(\rho, e)\,\boldsymbol{I}$ is the stress tensor given by the equation of state.

Artificial Viscosity

To facilitate shock wave propagation, the stress tensor is augmented by an artificial stress tensor

$$\boldsymbol{\sigma} = -EOS\,(\rho, e)\,\boldsymbol{I} + \boldsymbol{\sigma}_a$$

The artificial stress tensor is defined by a viscous coefficient, μ, and a symmetrized velocity gradient, $\epsilon(\boldsymbol{v})$, as

$$\boldsymbol{\sigma}_a = \mu\,\epsilon(\boldsymbol{v})\,, \qquad \epsilon(\boldsymbol{v}) \equiv {}^1\!/_2\,(\nabla\boldsymbol{v} + \boldsymbol{v}\nabla)$$

The coefficient μ is defined in terms of some measure of compression (or velocity jump)

- *Standard*: $\mu = \mu_{\text{std}} \equiv \rho\left(q_2 h^2\,|\Delta_s \boldsymbol{v}| + q_1\psi_0\psi_1 h c_s\right)$
- *Hyperviscosity*: $\mu_{\text{hyp}} \equiv q_3\rho h^{2k}\overline{|\Delta^k s|}, \quad \mu = \min\left(\mu_{\text{std}}, \mu_{\text{hyp}}\right)$

$$s = \left(\sqrt{\epsilon} : \epsilon\right) h_0^2\left(\frac{\det \boldsymbol{J}}{\det \boldsymbol{J}_0}\right)^{2/d}, \quad \Delta^k = k\text{-th order Laplacian}$$

The standard viscous term achieves shock capturing but remains low-order while the hyperviscous term is able to do both.

Finite Element Formulation of Hyperviscous Term

In implementing the hyperviscous term of the artificial stress tensors a Laplacian operator of the form $f = \Delta u$ must be discretized using finite elements. We proceed by formulating a weak form of the operator using a Galerkin method:

$$\int_\Omega f\,\phi_i\,\mathrm{d}\boldsymbol{x} = \int_\Omega \Delta u\,\phi_i\,\mathrm{d}\boldsymbol{x} = -\int_\Omega \nabla u\cdot\nabla\phi_i\,\mathrm{d}\boldsymbol{x} + \int_{\partial\Omega}(\nabla u\cdot\hat{\nu})\,\phi_i\,\mathrm{d}\boldsymbol{x}$$

where $\phi_i\,(\boldsymbol{x})$ corresponds to the basis function associated with the i-th degree of freedom, and all functions are expanded in terms of the basis functions as $f \approx \sum_j^{N_{\text{DOF}}} f_j\,\phi_j$. By replacing the expansions we get the linear system:

$$\boldsymbol{M}\boldsymbol{f} = \boldsymbol{S}\boldsymbol{u} \rightarrow \boldsymbol{f} = \boldsymbol{M}^{-1}\boldsymbol{S}\boldsymbol{u}$$

where the mass matrix, $\boldsymbol{M}$, and the stiffness matrix, $\boldsymbol{S}$, are given by

$$m_{ij} = \int_\Omega \phi_i\,\phi_j\,\mathrm{d}\boldsymbol{x}, \qquad s_{ij} = \int_\Omega \nabla\phi_i\cdot\nabla\phi_j\,\mathrm{d}\boldsymbol{x} + \int_{\partial\Omega}(\nabla\phi_j\cdot\hat{\nu})\,\phi_i\,\mathrm{d}\boldsymbol{x}$$

and thus the discrete Laplacian operator becomes $\Delta \approx \boldsymbol{M}^{-1}\boldsymbol{S}$. Higher order Laplacians are calculated by successive applications of this discrete operator.

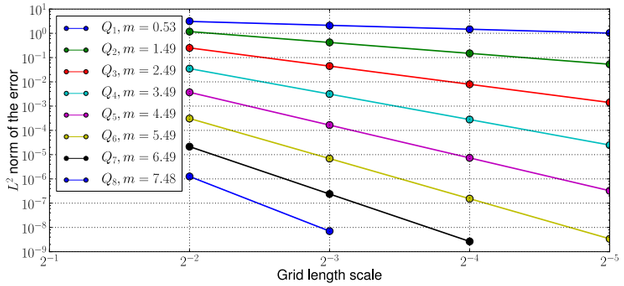

Figure 1: Convergence study of the discrete Laplacian operator applied to $f = {}^1\!/_c \sin(cx)\cos(cy)$ with exact solution $\Delta f = -2c\sin(cx)\cos(cy)$

Matrix-Free Laplacian

The k-th order Laplacian can be computed in discrete form using two operators:

- *Derivative matrix* (cheap, "low-order"): $\Delta^k f\,(\boldsymbol{x}) \approx \boldsymbol{D}_x^{2k}\boldsymbol{f} + \boldsymbol{D}_y^{2k}\boldsymbol{f}$
- *Variational form* (expensive, high-order): $\Delta^k f\,(\boldsymbol{x}) \approx \left(-\boldsymbol{M}^{-1}\boldsymbol{S}\right)^k\boldsymbol{f}$

We want an *assembly*-free form of $\boldsymbol{M}$ and $\boldsymbol{S}$ that is scalable and cheap to apply. For rectangular and hexahedral elements tensor products of matrices arising from 1D operators enable low-operation-count matrix-vector multiplications, $\mathcal{O}\left(p^{d+1}\right)$ operations:

$$\boldsymbol{M} \rightarrow m_{k_1,k_2,i_1,i_2} = m^1_{k_1,i_1} m^2_{k_2,i_2}$$

$$\boldsymbol{v} = \boldsymbol{M}\boldsymbol{u} \rightarrow v_{k_1,k_2} = m_{k_1,k_2,i_1,i_2} u_{i_1,i_2} = m^1_{k_1,i_1} u_{i_1,i_2} m^2_{k_2,i_2}$$

When running parallel jobs with this method exchanging information between element boundaries is the only communication needed.

Hydrodynamics Results for Smooth Solutions

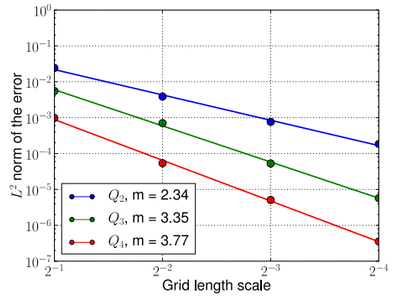

Figure 2: No artificial viscosity

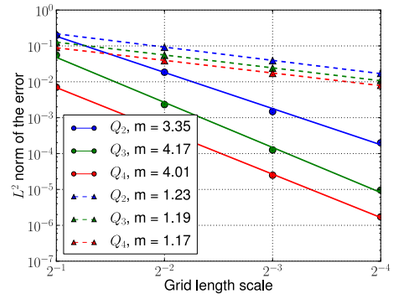

Figure 3: Standard (dashed) and hyper (solid) artificial viscosity

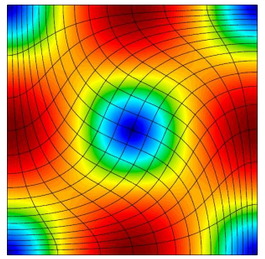

Figure 4: Taylor-Green

Using the Taylor-Green vortex we can study the convergence properties of the Lagrangian hydrodynamics equations. Our results confirm that the standard stress tensor is first order convergent while high-order convergence is achieved when the hyperviscous term is used.

Hydrodynamics Results for Non-Smooth Solutions

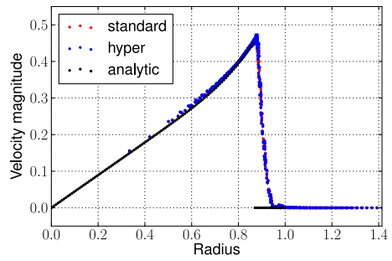

Figure 5: Radial results

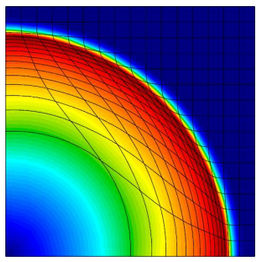

Figure 6: Sedov blast

Conclusions and Ongoing Work

In this work a high-order hyperviscous term is presented and shown to work for a range of problems with and without smooth solutions. An assembly-free form of the operators is currently being implemented and we expect to see an overall performance improvement of the running time due to the reduced operation count when applying 1D tensors. We also plan to use GPUs to accelerate even further the overall execution on high-performance computing systems.

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